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On the Hamiltonian $p^4 + V(r)$ (*).

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Summary. — As suggested by an extension of the supersymmetric Wess-Zumino model to higher dimensions we consider the eigenvalue problem for the Hamiltonian $p^4 + V(r)$, where V is either a δ function or the Coulomb potential (which happens to be the Green's function for the bilaplacian in five dimensions).

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1. – Introduction.

For several reasons, it is interesting to consider differential equations of order higher than the second. For example, they appear in discussions of

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

quantum gravity [1]. Also in an extension of supersymmetric Wess-Zumino model to higher dimensions, a generalized Klein-Gordon equation of the form

$$[\Box^{2^{d/2-2}} + m^{2^{d/2-2}}]\Psi = 0, \tag{1}$$

was obtained [2,3].

This equation, for $d = 4$, gives the usual Klein-Gordon equation and for $d = 6$ it gives

$$\Box\Box\Psi + m^4\Psi = 0. \tag{2}$$

Several arguments have been advanced to try theories in spaces of higher dimensions. Once they have been accepted into the game then, for the reasons mentioned above, higher-order equations might also be considered in it.

Therefore, it is justified to try to gain some experience about physical properties of systems obeying higher-order equations. We do not intend to take a mathematical point of view (see ref. [4]); our aim here is rather to discuss some examples that could be of physical interest.

To go from the higher-order Lagrangians to a Hamiltonian system would require a detailed analysis. In order to simplify matters we will assume the following equation for stationary states:

$$H\psi = E\psi \tag{3}$$

with

$$H = \nabla^2\nabla^2 + V(r), \tag{4}$$

and we will discuss this equation for some potentials $V(r)$ and for spherically symmetric solutions.

In sect. **2**, in order to illustrate the main line of the method, we revisit the usual hydrogen atom.

In sect. **3**, we discuss the case in which $V(r)$ is a δ function.

In sect. **4**, the potential $V(r) = -\alpha/r$ is considered. This is the Green's function of the bilaplacian operator in five dimensions.

In sect. **5**, boundary conditions are discussed. Finally, in two appendices we show briefly an equation of the fourth order with solutions similar to that of the usual harmonic oscillator and elaborate on the conditions of self-adjointness.

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(1) S. W. HAWKING: preprint University of Cambridge, Department of Applied Mathematics and Theoretical Physics (September 1985).
(2) C. G. BOLLINI and J. J. GIAMBIAGI: *Phys. Rev. D*, **32**, 3316 (1985).
(3) R. DELBURGO and V. B. PRASAD: *J. Phys. G*, **1**, 377 (1975).
(4) W. W. ZACHARY: *J. Math. Anal. Appl.*, **117**, 449 (1986).

2. – The hydrogen atom revisited.

In order to illustrate the method we shall follow for the fourth order, we first consider the usual second-order Schrödinger equation for the spherically symmetric solutions of the hydrogen atom

$$\left(-\nabla^2 - \frac{\alpha}{r}\right)\psi = E\psi, \tag{5}$$

which, with $\psi = \chi/r$, can be written as

$$\frac{d^2\chi}{dr^2} + \frac{\alpha}{r}\chi = -E\chi. \tag{6}$$

We shall use the Laplace transform (L)

$$\chi(r) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \exp[pr]\,\phi(p)\,dp, \qquad \phi(p) = \int\limits_0^\infty \exp[-pr]\,\chi(r)\,dr. \tag{7}$$

We look for a solution with physical boundary conditions

$$\chi(0) = 0, \qquad \dot\chi(0) = 1. \tag{8}$$

Taking into account that (ref. (5), p. 129)

$$L\left(\frac{1}{r}\chi(r)\right) = \int\limits_\pi^\infty \phi(p')\,dp', \tag{9}$$

we get for $\phi(p)$:

$$p^2\phi(p) + \alpha\int\limits_p^\infty \phi(p')\,dp' + E\phi(p) = 0 \tag{10}$$

and after taking the derivative,

$$(p^2 + E)\frac{d\phi}{dp} + (2p - \alpha)\,\phi(p) = 0 \tag{11}$$

which on integration leads to

$$\phi(p) = \frac{1}{p^2 + E}\left(\frac{p - i\sqrt{E}}{p + i\sqrt{E}}\right)^{\alpha/(2i\sqrt{E})}, \tag{12}$$

which is valid for any real E.

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(5) *Bateman Manuscript Project: Tables of Integral Transforms*, Vol. 1 (McGraw-Hill Book Co. Inc. New York, N.Y., 1954).

For $E<0$ it is convenient to write (12) in the form

$$\phi(p)=\frac{1}{p^2-|E|}\left(\frac{p-|E|^{1/2}}{p+|E|^{1/2}}\right)^{\alpha/(2|E|^{1/2})}, \tag{13}$$

where we used $E=-|E|$.

The Laplace antitransform (L^{-1}) of $(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}$ is (ref. (5), p. 238)

$$L^{-1}[(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}]=\frac{r^{-\beta_1-\beta_2-1}\exp[\lambda_1 r]}{\Gamma(-\beta_1-\beta_2)},\quad x_1F_1(-\beta_2;-\beta_2-\beta_1;(\lambda_2-\lambda_1)r). \tag{14}$$

To find out which values of E correspond to eigenfunctions we must look for the behavior of ${}_1F_1$ as $r\to\infty$. We have, for the asymptotic behaviour of ${}_1F_1$ (ref. (6), p. 278)

$${}_1F_1(-\beta_2;-\beta_2-\beta_1;2|E|^{1/2}r)=\frac{\Gamma(-\beta_2-\beta_1)}{\Gamma(-\beta_2)}(2|E|^{1/2}r)^{\beta_1}\exp[2|E|^{1/2}r], \tag{15}$$

where we used $\lambda_1=-|E|^{1/2}$ and $\lambda_2=|E|^{1/2}$. Thus, for $r\to\infty$,

$$\chi(r)\to(2|E|^{1/2})^{\beta_1}\frac{r^{-\beta_2-1}}{\Gamma(-\beta_2)}\exp[|E|^{1/2}r]. \tag{16}$$

This shows that $\chi(r)$ diverges unless we have $\beta_2=n$, where n is a nonnegative integer. That is, from (13),

$$\beta_2=\frac{\alpha}{2|E|^{1/2}}-1=n \tag{17}$$

or

$$E=-\frac{\alpha^2}{4(n+1)^2}\qquad \text{for } n=0,1,2,\ldots. \tag{18}$$

In short, the asymptotic behaviour can be directly obtained by noting that the behaviour for large r is dominated by the singularity $(p-\lambda)^{\lambda}$ which has the largest real part of λ.

In fact,

$$L^{-1}(p-\lambda)^{\beta}=\frac{\exp[\lambda r]}{\Gamma(-\beta)\,r^{\beta+1}}, \tag{19}$$

which leads formally to (17) and (18).

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(6) *Bateman Manuscript Project: Higher Trascendental Functions*, Vol. 1 (McGraw-Hill Book Co. Inc. New York, N.Y., 1953).

With positive E, the singularities appear on the imaginary axis (see (12)) and this leads to scattering states for any value of $E > 0$.

It is worth noting that eq. (11) gives information about the locations of singularities and energy eigenvalues even without knowing its explicit solutions. We assume that near a singularity λ, $\phi(p)$ has the form

$$\phi(p) = (p-\lambda)^\beta + O((p-\lambda)^{\beta+1}). \tag{20}$$

Replacing in (11) we have

$$[(p-\lambda)^2 + 2\lambda(p-\lambda) + \lambda^2 + E]\beta(p-\lambda)^{\beta-1} + 2[2(p-\lambda) + 2\lambda - \alpha](p-\lambda)^\beta = 0. \tag{21}$$

Disregarding $(p-\lambda)^{\beta+1}$ we get

$$\lambda^2 = -E, \qquad \lambda = \frac{\alpha}{2(\beta+1)}. \tag{22}$$

When the energy is positive, these singularities are located on the imaginary axis giving the scattering states. When the energy is negative, the singularities lay on the real axis at $\lambda = \pm |E|^{1/2}$. In order to avoid the singularity on the right-hand side plane, β should be chosen equal to a positive (or zero) integer leading to form (17).

Going back to (8) we want to point out that $\phi(p)$ goes to zero like $1/p^2$ when $|p| \to \infty$. As it has no poles on the right-hand side of the p-plane, for $r = 0$ one can close the integration path in (7) with a semicircle to the right. We then see that $\chi(0) = 0$, but $\dot{\chi}(0) \neq 0$.

3. – δ-function potential.

We now want to find the spherically symmetric solution of the equation

$$\nabla^2 \nabla^2 \psi - \alpha \delta^3(\mathbf{r}) \psi = E\psi. \tag{23}$$

We first look for a solution of (23) which outside the origin is a «free» equation. The boundary conditions at $r = 0$ are left open so as to able to adjust them to generate the δ-function potential.

We choose $\psi = \chi/r$ and obtain for $r \neq 0$

$$\frac{d^4\chi}{dr^4} = E\chi. \tag{24}$$

Taking the Laplace transform we get (ref. [5], p. 129)

$$(p^4 - E)\phi(p) = p^3\chi(0) + p^2\chi'(0) + p\chi''(0) + \chi'''(0). \tag{25}$$

If $\chi(0) \neq 0$, then ψ has a $1/r$ singularity. Since $\nabla^2(1/r) \simeq \delta(\boldsymbol{r})$ and $\nabla^2\nabla^2(1/r) \simeq \nabla^2\delta$ this leads to a singularity not contained in eqs. (23) or (24); so we must impose $\chi(0) = 0$, that is

$$\psi(0) = \chi'(0) . \tag{26}$$

Thus, in (25) we drop the p^3 term and obtain

$$\phi(p) = \frac{p^2\chi'(0) + p\chi''(0) + \chi'''(0)}{(p-\lambda_1)(p-\lambda_2)(p-\lambda_3)(p-\lambda_\psi)} , \tag{27}$$

where the λ_i are the four roots of the equation

$$\lambda_i^4 = E . \tag{28}$$

Let us first look for solutions with negative $E = -|E|$. Explicitly,

$$\begin{cases} \lambda_1 = \dfrac{1+i}{\sqrt{2}} |E|^{1/4} , & \lambda_2 = \lambda_1^* , \\ \lambda_3 = \dfrac{-1+i}{\sqrt{2}} |E|^{1/4} , & \lambda_4 = \lambda_3^* . \end{cases} \tag{29}$$

We have seen that singularities in the right-hand plane generate solutions which increase exponentially for large r. So we must eliminate them in eq. (27). These correspond to λ_1 and λ_2; then we choose the constants in such a way that

$$p^2\chi'(0) + p\chi''(0) + \chi'''(0) = \boldsymbol{C}(p-\lambda_1)(p-\lambda_2) , \tag{30}$$

which leads to

$$\chi'(0) = \boldsymbol{C} , \qquad \chi''(0) = -\boldsymbol{C}\sqrt{2}|E|^{1/4} , \qquad \chi'''(0) = \boldsymbol{C}|E|^{1/2} . \tag{31}$$

With these values we get

$$\phi(p) = \frac{\boldsymbol{C}}{(p-\lambda_3)(p-\lambda_4)} \tag{32}$$

whose anti-Laplace transform is (ref. [5], p. 229)

$$\chi(r) = \frac{\boldsymbol{C}}{\lambda_3 - \lambda_4}(\exp[\lambda_3 r] - \exp[\lambda_4 r]) . \tag{33}$$

That is,

$$\Psi(r) = \frac{\sqrt{2}\,\boldsymbol{C}}{|E|^{1/4} r} \exp\left[-\frac{|E|^{1/4}}{\sqrt{2}} r\right] \sin\frac{|E|^{1/4}}{\sqrt{2}} r. \tag{34}$$

Now we have

$$\nabla^2 \frac{\chi}{r} = \left(\nabla^2 \frac{1}{r}\right)\chi + \frac{1}{r}\frac{d^2\chi}{dr^2}. \tag{35}$$

The first term of (35) drops out as $\chi(0) = 0$ and we are left with

$$\nabla^2\left(\frac{\chi}{r}\right) = \frac{1}{r}\frac{d^2\chi}{dr^2}. \tag{36}$$

From here,

$$\nabla^2\nabla^2 \frac{\chi}{r} = \nabla^2\left(\frac{1}{r}\frac{d^2\chi}{dr^2}\right) = \left(\nabla^2 \frac{1}{r}\right)\frac{d^2\chi}{dr^2} + \frac{1}{r}\frac{d^4\chi}{dr^4}. \tag{37}$$

Comparing with (23) we have

$$\alpha = -4\pi \frac{\chi''(0)}{\psi(0)} = -4\pi \frac{\chi''(0)}{\chi'(0)} = -4\pi\sqrt{2}\,|E|^{1/4}, \tag{38}$$

from which

$$|E|^{1/4} = \frac{\alpha}{4\pi\sqrt{2}}, \tag{39}$$

which is possible only if $\alpha > 0$ (attractive potential). We also see that there is only one eigenvalue for each positive α:

$$E = -\frac{\alpha^4}{4(4\pi)^4}. \tag{40}$$

For $E > 0$, the singularities are at

$$\lambda_1 = E^{1/4}, \qquad \lambda_2 = -\lambda_1, \qquad \lambda_3 = i\lambda_1, \qquad \lambda_4 = -i\lambda_1. \tag{41}$$

Now, if in (27) we choose the quadratic form in the numerator so as to eliminate a couple of roots we arrive at an inconsistency. Namely, if we eliminate the two real roots or the two imaginary ones, then

$$\chi''(0) = -\boldsymbol{C}(\lambda_a + \lambda_b) = 0 \tag{42}$$

and so it is not possible to get a δ-function like in (36). On the other hand, if we choose to eliminate one real and one imaginary root we are led to a complex value for α.

We cannot avoid, however, eliminating the positive root (as it will, otherwise, imply an exponentially increasing function), so we choose

$$p^2\chi'(0) + p\chi''(0) + \chi'''(0) = \boldsymbol{C}(p-a)(p-\lambda_1)\,, \tag{43}$$

where the real parameter a should be different from λ_1 or λ_2. Now

$$\chi'(0) = \boldsymbol{C}\,, \qquad \chi''(0) = -\,\boldsymbol{C}(a+\lambda_1)\,, \qquad \chi'''(0) = \boldsymbol{C}a\lambda_1 \tag{44}$$

and

$$\phi(p) = \frac{\boldsymbol{C}(p-a)}{(p-\lambda_2)(p-\lambda_3)(p-\lambda_4)}\,, \tag{45}$$

whose Laplace transform is (ref. [5], p. 230):

$$\chi(r) = K\exp[i\sigma](\exp[-\lambda_1 r] - \exp[i\lambda_1 r]) + \\ + K\exp[i\delta](\exp[-\lambda_1 r] - \exp[-i\lambda_1 r])\,, \tag{46}$$

where σ and δ are arbitrary phases. From (46) we get

$$\alpha = -\,4\pi\frac{\chi''(0)}{\chi'(0)} = \frac{8\pi\lambda_1}{1-\operatorname{tg}(\sigma-\delta)/2}\,. \tag{47}$$

We see from (47) that the strength of the δ-function potential determines the phase difference of both terms in (46). We also see that the solution exists for any sign of α.

4. – The operator $p^4 - \alpha/r$.

Note that $1/r$ is the Green's function in five dimensions of the operator $\nabla^2\nabla^2$. We now deal with the equation

$$\nabla^2\nabla^2\psi - \frac{\alpha}{r}\psi = E\psi\,. \tag{48}$$

With $\psi = \chi/r$ we obtain

$$\frac{d^4\chi}{dr^4} - \frac{\alpha}{r}\chi = E\chi \tag{49}$$

with $\chi(0) = \chi''(0) = 0$ to avoid δ-functions at the origin (see sect. 3.).

For the time being, and for reasons of simplicity, we also take $\chi'(0) = 0$. With these initial conditions the Laplace transform of (49) is (after taking a derivative d/dp)

$$(p^4 - E)\frac{d\phi}{dp} + (4p^3 + \alpha)\phi = 0 . \tag{50}$$

Compare with (11) and note the differences in the sign of E and α. To get a qualitative idea of the problem we follow the analysis used at the end of sect. 2. We assume

$$\phi = (p - \lambda)^\beta + O((p - \lambda)^{\beta+1}) \tag{51}$$

and substitute in (50)

$$(p^4 - E)\beta(p - \lambda)^{\beta-1} + (4p^3 + \alpha)(p - \lambda)^{\beta-1} = 0 . \tag{52}$$

So, near $p \approx \lambda$ we obtain the conditions

$$\lambda^4 = E$$

and

$$4\lambda^3 + \alpha + 4\lambda^3\beta = 0 , \tag{53}$$

i.e.

$$\lambda^3 = -\frac{\alpha}{4(\beta + 1)} . \tag{54}$$

Now, if $E < 0$ we have

$$\lambda_1 = \frac{1+i}{\sqrt{2}}|E|^{1/4}, \quad \lambda_2 = \lambda_1^*, \quad \lambda_3 = \frac{-1+i}{\sqrt{2}}|E|^{1/4}, \quad \lambda_4 = \lambda_3^* . \tag{55}$$

It is verified that λ_i^3 is another root λ_j which leads through (54) to a complex value of β. Then the exponential growth of $\chi(r)$ for $r \to \infty$ cannot be avoided. So there is no solution for eq. (50) with negative E and normalizable $\chi(r)$ with the assumed initial conditions. On the other hand, if E is positive we have

$$\lambda_1 = E^{1/4}, \quad \lambda_2 = -\lambda_1 , \quad \lambda_3 = iE^{1/4}, \quad \lambda_4 = -\lambda_3 . \tag{56}$$

The dominant singularity corresponds to λ_1 (largest real part). This singularity can be avoided only if $\alpha < 0$ (see (54)) (repulsive potential) in which case eq. (54)

gives

$$E^{3/4} = \frac{|\alpha|}{4(\beta+1)}, \tag{57}$$

and choosing β integer (see discussion in sect. 2)

$$E_n = \left(\frac{|\alpha|}{4(n+1)}\right)^{4/3}. \tag{58}$$

However, after this choice we still have λ_3 and λ_4 on the imaginary axis and the corresponding asymptotic behaviour is now dominated by the waves $\exp[\lambda_3 r]$ and $\exp[\lambda_4 r]$. In the one-dimensional case this corresponds to the states of «total reflexion» (see ref. (7)).

As a matter of fact we can write the explicit solution of (50):

$$\phi(p) = \boldsymbol{C}(p-\lambda_1)^{\beta_1}(p-\lambda_2)^{\beta_2}(p-\lambda_3)^{\beta_3}(p-\lambda_4)^{\beta_4}, \tag{59}$$

where

$$\beta_i = -\frac{\alpha\lambda_i}{4E} - 1. \tag{60}$$

The inverse Laplace transform is (ref. (1), p. 238)

$$\chi(r) \simeq r^3 \Phi_2(-\beta_1, -\beta_2, -\beta_3, -\beta_4; 4; \lambda_1 r, \lambda_2 r, \lambda_3 r, \lambda_4 r). \tag{61}$$

where Φ_2 is defined in ref. (6) (p. 235). By studying the asymptotic behaviour of Φ_2, which is cumbersome and uninteresting, we arrive at the same conclusions already mentioned.

Formula (59) and (60) together with (55) and (56) show that as there are no poles on the right-hand side in (7) we can close the integration by a semicircle on the right, giving $\chi(0) = 0$. The same argument holds for $\chi'(0) = 0$ and $\chi''(0) = 0$ but not for $\chi'''(0)$. The reason is that the integrand vanishes like $1/p^4$.

5. – Modified boundary conditions.

The condition $\chi(0) = 0$ has to be imposed to avoid a $\nabla^2\delta(r)$ singularity. A similar argument is valid for $\chi''(0) = 0$ which otherwise leads to δ singularities. On the other hand, $\chi'(0)$ and $\chi'''(0)$ should be left as arbitrary constants. In the previous section we choose $\chi'(0) = 0$ for simplicity reasons. We now drop this requirement.

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(7) C. G. Bollini and J. J. Giambiagi: *Nuovo Cimento A*, 98, 151 (1987).

Taking into account eq. (25) we now get instead of (50) the inhomogeneous equation

(62) $$(p^4 - E)\frac{d\phi}{dp} + (4p^3 + \alpha)\phi = 2\chi'(0)p = 2Ap\,.$$

The general solution of (62) is (ref. ([8]), p. 16)

(63) $$\phi(p) = \exp[-F(p)]\left(\eta + \int_\varepsilon^p dx\, g(x)\exp[F(x)]\right),$$

where

(64) $$F(p) = \int_\varepsilon^p \frac{4x^3 + \alpha}{x^4 - E}dx \quad \text{and} \quad g(p) = \frac{2Ap}{p^4 - E}\,.$$

However eq. (63) is only a formal solution. We can obtain an eigenvalue condition as follows. For negative energy $E = -\varepsilon^4$ and using the transformation $x = r\varepsilon$, eq. (49) can be written in the form

(65) $$\frac{d^4\chi}{dx^4} - \left(\frac{\varkappa}{x} - 1\right)\chi = 0\,,$$

where $\varkappa = \alpha/\varepsilon^3$. In the p-space it becomes an integral equation

(66) $$(p^4 + 1)\phi(p) - \varkappa\int_p^\infty \phi(p')\,dp' = Ap + B\,,$$

where $A = \chi'(0)$ and $B = \chi'''(0)$.

Since $\phi(p)$ must be regular at $p = \lambda_1$ and λ_2 (where $\lambda_{1,2} = (1 \pm i)/\sqrt{2}$) in order to get the correct behaviour of $\chi(r)$ for $r \to \infty$ (see sect. 2), we have from eq. (66):

(67) $$A\lambda_1^2 + B + \varkappa\int_{\lambda_1}^\infty \phi(p')\,dp' = 0\,,$$

(68) $$A\lambda_2^2 + B + \varkappa\int_{\lambda_2}^\infty \phi(p)\,dp' = 0\,,$$

and substracting we obtain the eigenvalue equation we were looking for

(69) $$2Ai + \varkappa\int_{\lambda_1}^{\lambda_2} \phi(p')\,dp' = 0\,,$$

where we can put the normalization constant $A = 1$.

([8]) E. KAMKE: *Differentialgleichungen* (Chelsea Publ. Co., 1971).

Numerical integration of (62) together with (69) yields for the ground state

$$E = -0.3626\,\alpha^{4/3}$$

and for the first excited state

$$E' = -0.114\,\alpha^{4/3}.$$

6. – Discussion.

The bilaplacian (p^4) equation has a peculiar behaviour when compared with the usual p^2 equation. The attractive δ-function potential has no negative eigenvalue, no bound state, in the p^2 case, while it has always one and only one for the p^4 case.

Furthermore, we get a solution for any positive value of E and for any sign of coupling constant α. It is a combination of imaginary exponentials plus exponentially decreasing functions as shown explicitly in eq. (46). In order to produce a δ-function behaviour at the origin it is essential to have $\chi''(0) \neq 0$.

The potential α/r is more involved. For the initial conditions $\chi(0) = \chi'(0) = \chi''(0) = 0$ there are no negative eigenvalues. On the other hand, if E is positive, with

$$E_n = \left(\frac{|\alpha|}{4(n+1)}\right)^{4/3} \tag{65}$$

(a discrete set of infinite eigenvalues!) then, the solutions behave like incoming and outgoing waves. They correspond to the solutions for «total reflexion» discussed in ref. (7). For any other values of $E > 0$ one cannot avoid the exponential increase for $r \to \infty$.

Using a variation method, Perez (9) has proved that there are infinite negative eigenvalues provided $\chi'(0) \neq 0$.

* * *

We are indebted to Dr. F. Perez for encouraging discussions.

APPENDIX A

We want to mention a particular example of a fourth-order self-adjoint equation whose eigensolutions are similar to those of the second-order harmonic

(9) F. PEREZ: Inst. de Fisica, Univ. de São Paulo, private communication.

oscillator:

$$\frac{d^4\phi}{dr^4} - 2\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) + r^4\phi = P\phi = E\phi . \tag{A.1}$$

Defining

$$y_n = r^n \exp[-r^2/2] , \tag{A.2}$$

we easily obtain

$$Py_n = [4(n^2+n)+3]\, y_n - 2n(2n^2-3n+1)\, y_{n-2} + n(n-1)(n-2)(n-3)\, y_{n-4} , \tag{A.3}$$

so we have

$$Py_n = \sum_{m=1}^{n} a_n^m y_m , \tag{A.4}$$

where

$$\begin{cases} a_n^n = E_n = 4n(n+1)+3 , \\ a_n^{n-2} = -2n(2n^2-3n+1) , \\ a_n^{n-4} = \dfrac{n!}{(n-4)!} . \end{cases} \tag{A.5}$$

We call ϕ_n the solution of

$$P\phi_n = E_n \phi_n , \qquad \text{where } E_n = 4n(n+1)+3 . \tag{A.6}$$

With

$$\phi_n = \sum_{l=1}^{n} A_n^l y_l , \tag{A.7}$$

we have

$$P\phi_n = \sum_{l=1}^{n} A_n^l \sum_{m=1}^{l} a_l^m y_m = E_n \sum_{m=1}^{n} A_n^m y_m . \tag{A.8}$$

Then,

$$\sum_{l=1}^{n} \left\{ \sum_{s=l}^{n} A_n^s a_s^l - E_n A_n^l \right\} y_s = 0 . \tag{A.9}$$

For $l = n$ we obtain

$$A_n^n a_n^n - E_n A_n^n = 0 \qquad \text{(identity)} \tag{A.10}$$

and thus

$$E_n = a_n^n \tag{A.11}$$

and we can set arbitrarily $A_n^n = 1$. For $l = n-2$ we get

$$A_n^{n-2} = \frac{a_n^{n-2}}{E_n - E_{n-2}}; \tag{A.12}$$

for $l = n-4$,

$$A_n^{n-4}(E_n - E_{n-4}) = a_n^{n-4} + A_n^{n-2} a_{n-2}^{n-4}, \tag{A.13}$$

and so on.

Appendix B

The conditions at $r=0$ which must be imposed on any two arbitrary eigenfunctions χ_1, χ_2 of the Hamiltonian to secure the hermiticity of the operator d^4/dr^4 are

$$\chi_1(0)\chi_2'''(0) - \chi_1'''(0)\chi_2(0) + \chi_1''(0)\chi_2'(0) - \chi_1'(0)\chi_2''(0) = 0. \tag{B.1}$$

In the case of the δ-function we put $\chi_n(0) = 0$, and the first two terms of (B.1) vanish. The remaining two terms cancel each other in virtue of the conditions (38) and (47). So, it is not necessary to have $\chi''(0) = 0$.

In the case of the Coulomb potential (B.1) is automatically satisfied by imposing the physical conditions $\chi(0) = \chi''(0) = 0$.

● Riassunto (*)

Come è suggerito da un'estensione del modello supersimmetrico di Wess-Zumino a dimensioni superiori, si considera il problema dell'autovalore per l'hamiltoniana $p^4 + V(r)$, dove V è una funzione δ o il potenziale di Coulomb (che si dà il caso sia una funzione di Green per il bilaplaciano a cinque dimensioni).

(*) *Traduzione a cura della Redazione.*

Резюме не получено.